

Unsolvable Problems

Part One

Outline for Today

- ***Self-Reference Revisited***
 - Programs that compute on themselves.
- ***Self-Defeating Objects***
 - Objects “too powerful” to exist.
- ***The Fortune Teller***
 - Can you escape your fate?
- ***Why Do Programs Loop?***
 - ... and can we eliminate loops?
- ***Undecidable Problems***
 - Something beyond the reach of algorithms.

Recap from Last Time

R and RE

- A language L is **recognizable** if there is a TM M with the following property:

$$\forall w \in \Sigma^*. (M \text{ accepts } w \leftrightarrow w \in L).$$

- That is, for any string w :
 - If $w \in L$, then M accepts w .
 - If $w \notin L$, then M does not accept w .
 - **What does this mean?**
- This is a “weak” notion of solving a problem.
- The class **RE** consists of all the recognizable languages.

R and RE

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$$\forall w \in \Sigma^*. (M \text{ accepts } w \leftrightarrow w \in L).$$

- That is, for any string w :
 - If $w \in L$, then M accepts w .
 - If $w \notin L$, then M does not accept w .
 - It might reject w , or it might loop on w .
- This is a “weak” notion of solving a problem.
- The class **RE** consists of all the recognizable languages.

R and RE

- A language L is **decidable** if there is a TM M with the following properties:

$\forall w \in \Sigma^*. (M \text{ accepts } w \leftrightarrow w \in L)$.

M halts on all inputs.
- That is, for any string w :
 - If $w \in L$, then M accepts w .
 - If $w \notin L$, then M rejects w .
- This is a “strong” notion of solving a problem.
- The class **R** consists of all the decidable languages.

The Universal TM

- The ***universal Turing machine***, denoted U_{TM} , is a TM with the following behavior: when run on a string $\langle M, w \rangle$, where M is a TM and w is a string, U_{TM} will
 - ... accept $\langle M, w \rangle$ if M accepts w ,
 - ... reject $\langle M, w \rangle$ if M rejects w , and
 - ... loop on $\langle M, w \rangle$ if M loops on w .
- A_{TM} is the language recognized by the universal TM. This is the language
$$A_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$$
- U_{TM} is a **??? for A_{TM}** .

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- U_{TM} is a **recognizer** for A_{TM} .

Self-Referential Programs

- Computing devices can compute on their own source code:
Theorem: It is possible to construct TMs that perform arbitrary computations on their own source code.
- This allows us to write programs that work on their own source code.

```
void cormorant() {
    string me = /* source code of
                  * cormorant
                  */;
    cout << me << endl;
}
```

```
bool curlew(string input) {
    string me = /* source code of
                  * curlew
                  */;
    return input == me;
}
```

```
int avocet() {
    string me = /* source code of
                  * avocet
                  */;
    int result = 0;
    for (char ch: me) {
        if (ch == 'a') result++;
    }
    return result;
}
```

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What do each of these pieces of code do?

New Stuff!

Part One: Self-Defeating Objects

A ***self-defeating object*** is an object whose essential properties ensure it doesn't exist.

Question: Why is there no largest integer?

Answer: Because if n is the largest integer, what happens when we look at $n+1$?

Self-Defeating Objects

Theorem: There is no largest integer.

Proof sketch: Suppose for the sake of contradiction that there is a largest integer. Call that integer n .

Consider the integer $n+1$.

Notice that $n < n+1$.

But then n isn't the largest integer.

Contradiction! ■-ish

Self-Defeating Objects

Theorem: There is no largest integer.

Proof sketch: Suppose for the sake of contradiction that there is a largest integer. Call that integer n .

Consider the integer $n+1$.

Notice that $n < n+1$.

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Contradiction! ■-ish

We're using n to construct something that undermines n , hence the term "self-defeating."

An Important Detail

Careful – we're assuming what we're trying to prove!

Claim: There is a largest integer.

Proof: Assume x is the largest integer. }

Notice that $x > x - 1$.

So there's no contradiction. ■-ish }

How do we know there's no contradiction? We just checked one case.

Self-Defeating Objects

- If you can show

$$x \text{ exists} \rightarrow \perp$$

then you know that x doesn't exist. (This is a proof by contradiction.)

- If you can show

$$x \text{ exists} \rightarrow \top$$

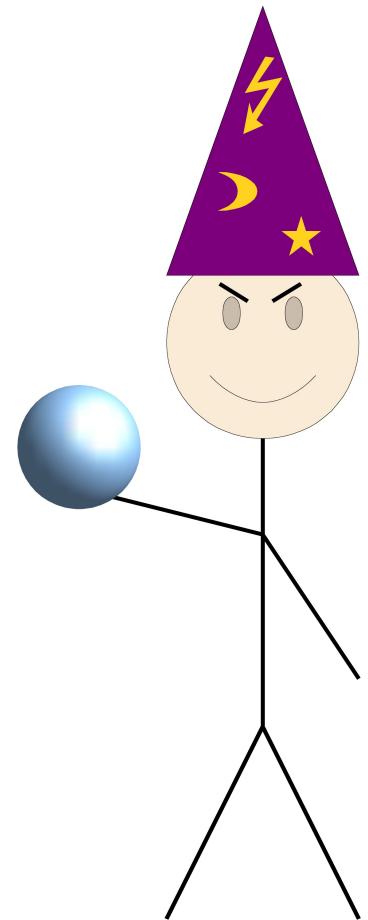
you cannot conclude that x exists. (This is not a valid proof technique.)

Part Two: The Fortune Teller

The Fortune Teller

- A fortune teller appears who claims they can see into the future.
- For a nominal fee, the fortune teller will tell you anything you want to know about the future.
- Of course, the fortune teller is a lying liar who lies. No one can see the future!
- The fortune teller makes a living taking money from unsuspecting townsfolk.

Someone needs to put an end to this!



The Fortune Teller

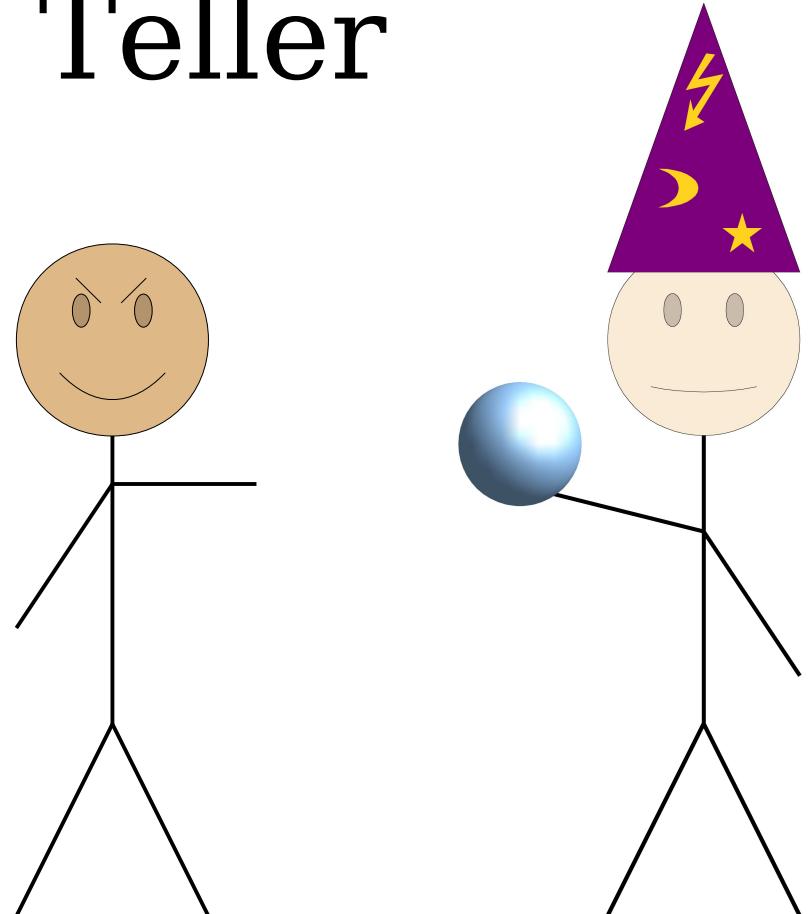
- One day, a trickster arrives. The trickster wants to expose that the fortune teller is a fraud.
- The trickster says the following:

“I have a yes/no question about the future. But before I ask my question, let’s talk payment.

If you answer ‘yes,’ then I’ll pay you \$42.

If you answer ‘no,’ then I’ll pay you \$137.”

- The fortune teller thinks for a moment, then agrees.



Trickster pays \$42 if the fortune teller answers “yes.”

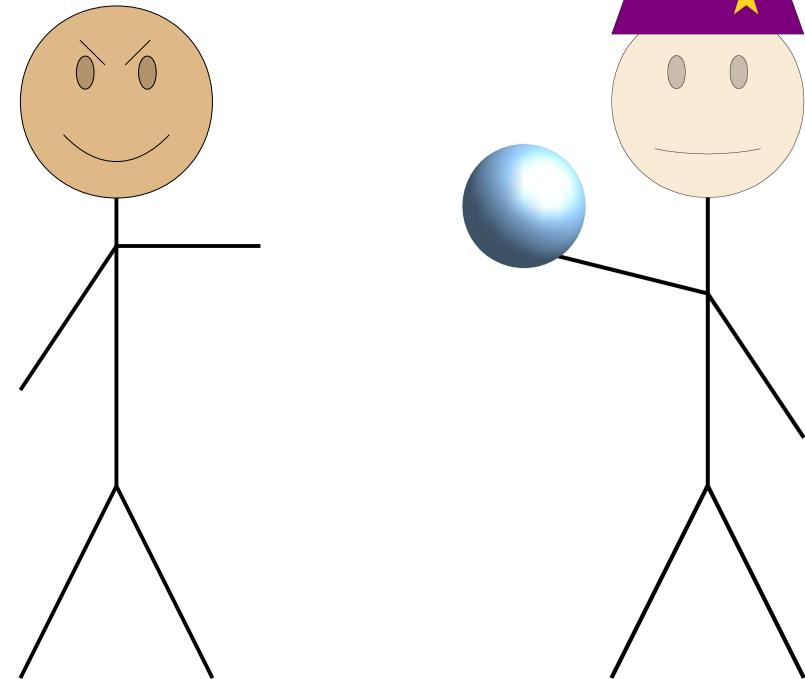
Trickster pays \$137 if the fortune teller answers “no.”

The Fortune Teller

- The trickster then asks this question:

“Am I going to pay you \$137?”

- The fortune teller is trapped!
- Why?



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Trickster pays \$42 if the fortune teller answers “yes.”

Trickster pays \$137 if the fortune teller answers “no.”

The Fortune Teller

- The payment scheme the fortune teller agreed to means

Fortune Teller Says Yes \leftrightarrow ***Trickster Pays \$42.***

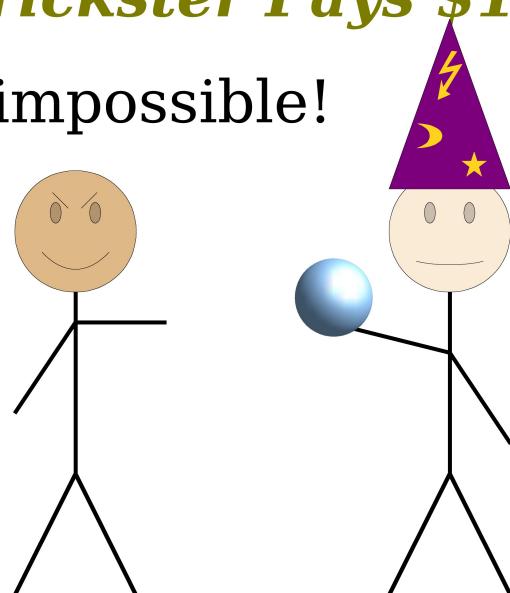
- The trickster's question to the fortune teller means

Fortune Teller Says Yes \leftrightarrow ***Trickster Pays \$137.***

- Putting this together, we get

Trickster Pays \$137 \leftrightarrow ***Trickster Pays \$42.***

- This is impossible!

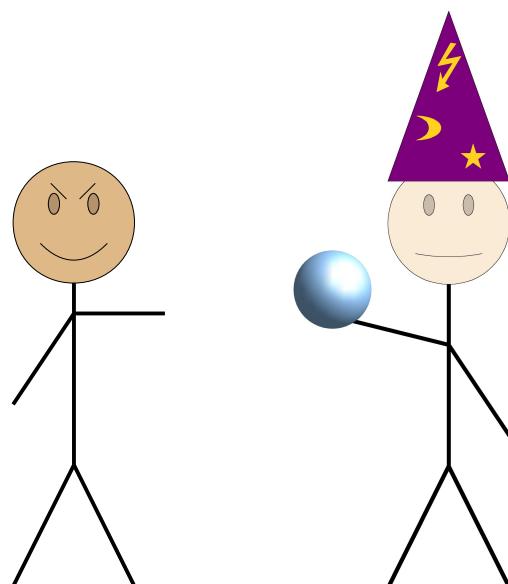


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The Fortune Teller

- The fortune teller is a self-defeating object.
- The trickster's strategy is to couple the fortune teller's behavior to what the future holds.
 - The trickster's behavior is chosen in advance to make the fortune teller's answer wrong.
- Therefore, the fortune teller can't answer all questions about all people in the future.



Trickster pays \$42 if the fortune teller answers “yes.”

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Part Three: Why Do Programs Loop?

Thoughts on Loops

- In practice, the programs we write sometimes go into infinite loops.
- In Theoryland, Turing machines are allowed to loop. This happens if they don't accept and don't reject.
- **Question:** Why are infinite loops possible?
- Or rather: are infinite loops an inherent part of computation, or are they some weird sort of “accident” in how we program computers?

Thoughts on Loops

- **[Major] Theorem:** The language A_{TM} is recognizable, but undecidable.
 - There's a *recognizer* for A_{TM} (specifically, the universal Turing machine U_{TM}).
 - It is impossible to build a *decider* for this language.
- Stated differently, there's a program we can write (a universal TM) that *has* to loop infinitely on some inputs.
- **Goal:** Prove this theorem, and explore its theoretical and philosophical implications.

A_{TM} Revisited

- As a refresher, the language A_{TM} is
 $A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$.
- The universal TM U_{TM} has the following behavior when given as input a TM M and a string w :
 - If M accepts w , then U_{TM} accepts $\langle M, w \rangle$.
 - If M rejects w , then U_{TM} rejects $\langle M, w \rangle$.
 - If M loops on w , then U_{TM} loops on $\langle M, w \rangle$.
- U_{TM} is a recognizer for A_{TM} , but because of that last case it's not a decider for A_{TM} .

A_{TM} Revisited

- As a refresher, the language A_{TM} is
 $A_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$.
- Given a TM M and a string w , a decider D for A_{TM} would need to have this behavior:
 - If M accepts w , then $D \quad ? \quad \langle M, w \rangle$.
 - If M rejects w , then $D \quad ? \quad \langle M, w \rangle$.
 - If M loops on w , then $D \quad ? \quad \langle M, w \rangle$.

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 - If M loops on w , then $D \quad ? \quad \langle M, w \rangle$.
- This is basically the same set of requirements as U_{TM} , except for what happens if M loops on w .
- Our goal is to prove that there is no way to build a program that meets these requirements.

A_{TM} Revisited

- As a refresher, the language A_{TM} is
 $A_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$.
- Given a TM M and a string w , a decider D for A_{TM} would need to have this behavior:
 - If M accepts w , then D accepts $\langle M, w \rangle$.
 - If M rejects w , then D rejects $\langle M, w \rangle$.
 - If M loops on w , then D rejects $\langle M, w \rangle$.
- This is basically the same set of requirements as U_{TM} , except for what happens if M loops on w .
- Our goal is to prove that there is no way to build a program that meets these requirements.

A_{TM} Revisited

- We can envision a decider for A_{TM} as a function

```
bool willAccept(string fn, string input)
```

that takes as input the source code of a function (`fn`) and a string representing an input to that function (`input`).

- It then does the following:
 - If $fn(input)$ returns true, `willAccept(fn, input)` returns true.
 - If $fn(input)$ returns false, `willAccept(fn, input)` returns false.
 - If $fn(input)$ loops, then `willAccept(fn, input)` returns false.
- We're going to show it's impossible to write a function that actually does this. But for now, let's just explore what such a decider would do.

```
function = "bool f(string input) {  
    if (input == "") return false;  
    return input[0] == 'a';  
}";
```

```
input = "abbababba";
```

```
willAccept(function, input) = ?
```

```
function = "bool h(string input) {  
    int n = input.length();  
    while (n > 1) {  
        if (n % 2 == 0) n /= 2;  
        else n = 3*n + 1;  
    }  
    return true;  
}";
```

```
input = /* 10137 a's */;
```

```
willAccept(function, input) = ?
```

```
function = "bool g(string input) {  
    while (true) {  
        input += input;  
    }  
}";
```

```
input = "yay! ";
```

```
willAccept(function, input) = ?
```

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For each of these instances, what does
`willAccept(function, input)` return?

Deciding A_{TM}

- Surprising fact: until 2019, no one knew whether there were integers x , y , and z where

$$x^3 + y^3 + z^3 = 33.$$

- A heavily optimized computer search found this answer:

$$\begin{aligned}x &= 8,866,128,975,287,528 \\y &= -8,778,405,442,862,239 \\z &= -2,736,111,468,807,040\end{aligned}$$

- As of November 2024, no one knows whether there are integers x , y , and z where

$$x^3 + y^3 + z^3 = 114.$$

Deciding A_{TM}

- Consider the language

$$L = \{ \mathbf{a}^n \mid \exists x \in \mathbb{Z}. \exists y \in \mathbb{Z}. \exists z \in \mathbb{Z}. x^3 + y^3 + z^3 = n \}$$

- Here's code for a recognizer to see whether such a triple exists:

```
bool hasTriple(int n) {
    for (int max = 0; ; max++)
        for (int x = -max; x <= max; x++)
            for (int y = -max; y <= max; y++)
                for (int z = -max; z <= max; z++)
                    if (x*x*x + y*y*y + z*z*z == n)
                        return true;
}
```

- Imagine calling `willAccept(/* hasTriple code */ , 114).`
 - If such a triple exists, `willAccept` returns true.
 - If no such triple exists, `willAccept` returns false.
- **Key Intuition:** However `willAccept` is implemented, it has to be clever enough to resolve open problems in mathematics!

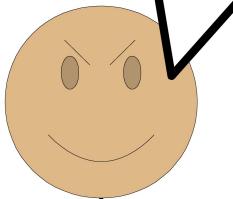
Why is A_{TM} Hard?

- **Intuition:** A decider for A_{TM} would be able to...
 - ... determine whether the hailstone sequence terminates for any input. (Write a recognizer that runs the hailstone sequence, then feed it into the decider for A_{TM} .)
 - ... see if any number is the sum of three cubes. (Write a recognizer that tries all infinitely many triples of integers, then feed it into the decider for A_{TM} .)
 - ... and much, much more.
- In other words, this seemingly simple problem of “is this program going to terminate?” accidentally scoops up a bunch of other seemingly harder problems.

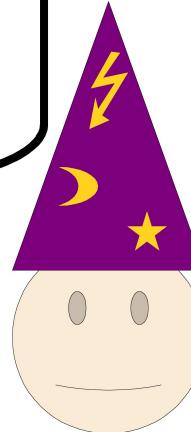
Part Four: Putting It All Together

To Recap

- We're assuming that, somehow, someone wrote a function
`bool willAccept(string function, string input);`
that takes the code of a function and an input to that function, then
 - returns true if `function(input)` returns true, and
 - returns false if `function(input)` doesn't return true.
- **Goal:** Show that this decider is “self-defeating;” its power is so great that it undermines itself.
- **Idea:** Convert the fortune teller story into a program.



*Am I going
to pay you
\$137?*



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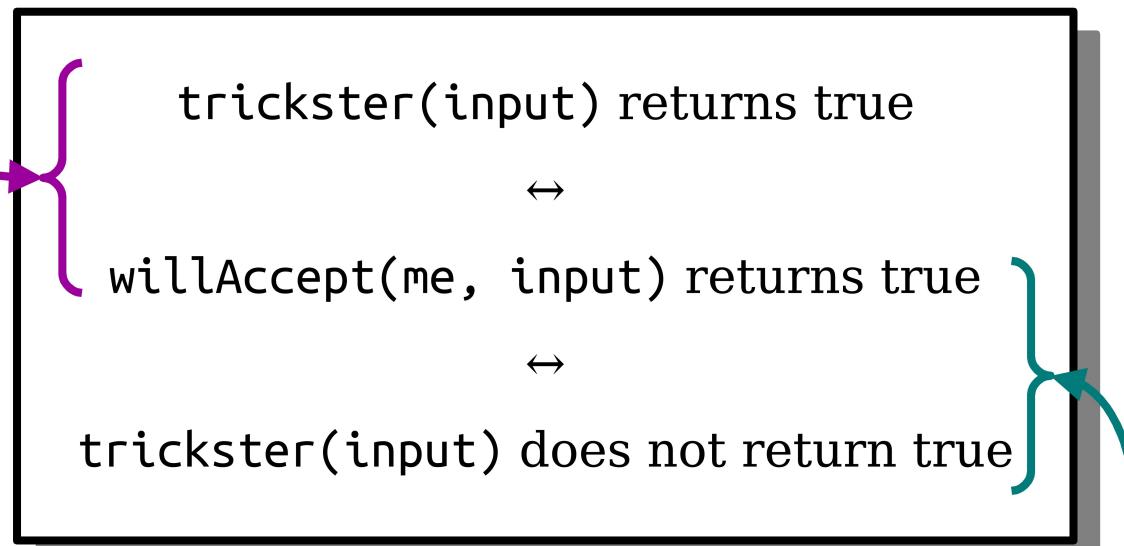
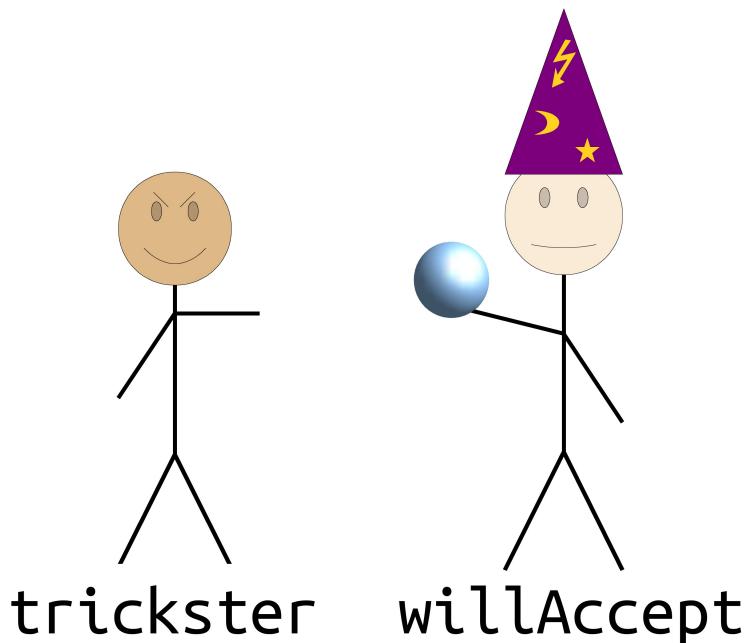
```

bool willAccept(string function, string input) {
    // Returns true if function(input) returns true.
    // Returns false otherwise.
}

bool trickster(string input) {
    string me = /* source code of trickster */;
    return !willAccept(me, input);
}

```

A decider for A_{TM} has to have this behavior.



Because of how we wrote trickster.

```
bool willAccept(string function, string input) {  
    // Returns true if function(input) returns true.  
    // Returns false otherwise.  
}
```

A self-defeating object.

```
bool trickster(string input) {  
    string me = /* source code of trickster */;  
    return !willAccept(me, input);  
}
```

Using that object against itself.

```
bool willAccept(string function, string input) {  
    // Returns true if function(input) returns true.  
    // Returns false otherwise.  
}
```

"The largest integer n ."

```
bool trickster(string input) {  
    string me = /* source code of trickster */;  
    return !willAccept(me, input);  
}
```

"The integer $n + 1$."

Theorem: There is no largest integer.

Proof sketch: Suppose for the sake of contradiction that there is a largest integer. Call that integer n .

Consider the integer $n+1$.

Notice that $n < n+1$.

But then n isn't the largest integer.

Contradiction! ■-ish

Theorem: $A_{TM} \notin \mathbf{R}$.

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Proof: By contradiction; assume that $A_{\text{TM}} \in \mathbf{R}$. Then there is a decider D for A_{TM} . We can represent D as a function

```
bool willAccept(string function, string w);
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that takes in the source code of a function $function$ and a string w , then returns true if $function(w)$ returns true and returns false otherwise.

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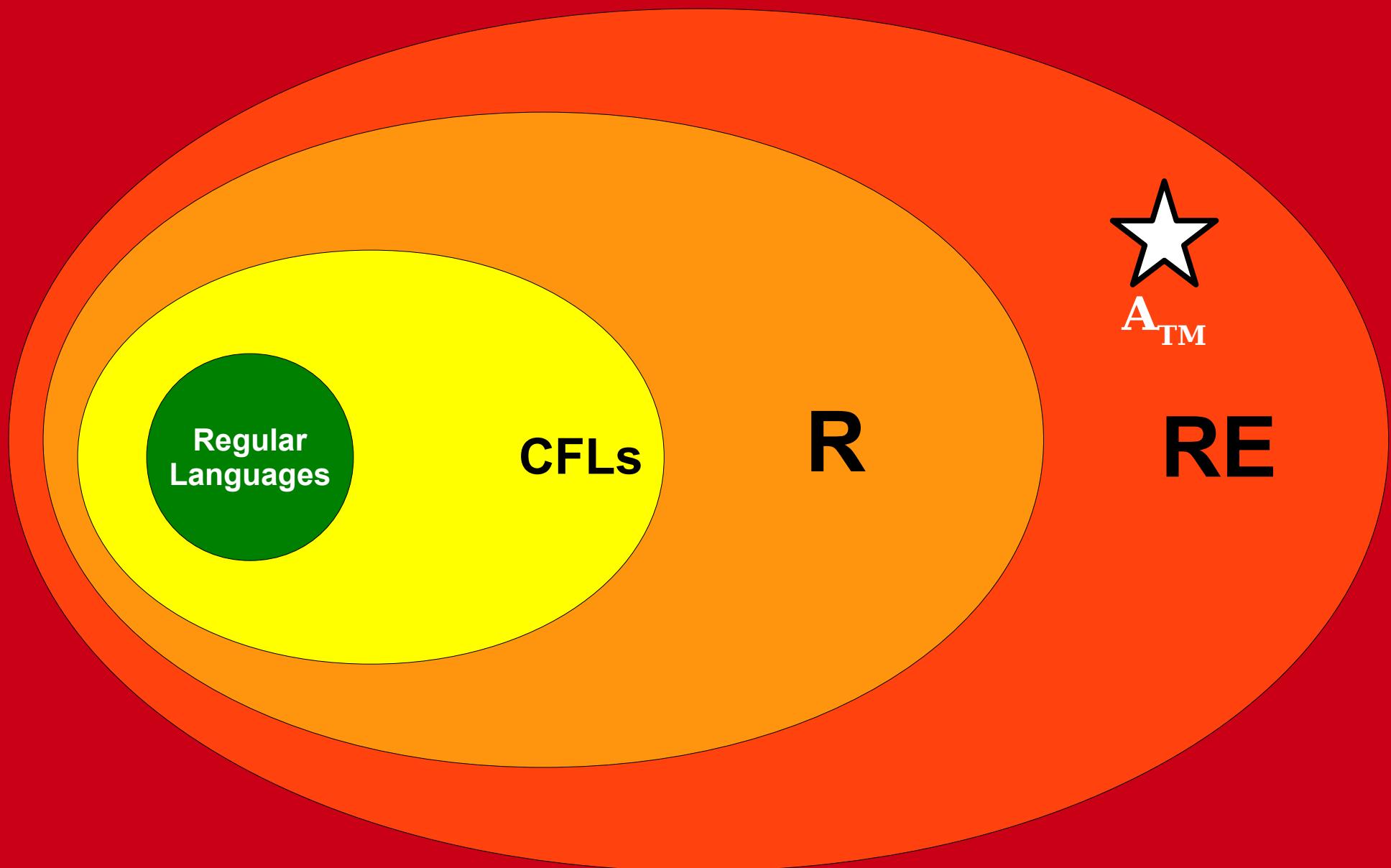
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All Languages

What Does This Mean?

- In one fell swoop, we've proven that
 - A_{TM} is **undecidable**; there is no general algorithm that can determine whether a TM will accept a string.
 - $R \neq \text{RE}$, because $A_{\text{TM}} \notin R$ but $A_{\text{TM}} \in \text{RE}$.
- What do these three statements really mean? As in, why should you care?

$$A_{\text{TM}} \notin R$$

- What exactly does it mean for A_{TM} to be undecidable?

Intuition: The only general way to find out what a program will do is to run it.

- As you'll see, this means that it's provably impossible for computers to be able to answer most questions about what a program will do.

$$A_{\text{TM}} \notin \mathbf{R}$$

- At a more fundamental level, the existence of undecidable problems tells us the following:

There is a difference between what is true and what we can discover is true.

- Given a TM M and a string w , one of these two statements is true:

M accepts w

M does not accept w

But since A_{TM} is undecidable, there is no algorithm that can always determine which of these statements is true!

$$\mathbf{R} \neq \mathbf{RE}$$

- Because $\mathbf{R} \neq \mathbf{RE}$, there is a difference between decidability and recognizability:

In some sense, it is fundamentally harder to solve a problem than it is to check an answer.

- There are problems where, when the answer is “yes,” you can confirm it (run a recognizer), but where if you don’t have the answer, you can’t come up with it in a mechanical way (build a decider).

Next Time

- ***Why All This Matters***
 - Important, practical, undecidable problems.
- ***Intuiting RE***
 - What exactly is the class **RE** all about?
- ***Verifiers***
 - A totally different perspective on problem solving.
- ***Beyond RE***
 - Finding an impossible problem using very familiar techniques.